APPENDIX

TRANSFORMATION OF ECI (CIS, EPOCH J2000.0) COORDINATES TO WGS 84 (CTS, ECEF) COORDINATES

1. General

This Appendix has two purposes. These are:

- To provide additional details on the mathematical relationship between the Conventional Inertial System (CIS; FK5 System, Epoch J2000.0), the Instantaneous Terrestrial System (ITS), and the Conventional Terrestrial System (CTS) as they relate to World Geodetic System 1984 (WGS 84).

- To discuss and provide the numerical data needed to transform position and velocity components ($X, Y, Z; \dot{X}, \dot{Y}, \dot{Z}$) from the CIS, or Earth-Centered-Inertial (ECI) Coordinate System, to the WGS 84 Earth-Centered-Earth-Fixed (ECEF) Coordinate System, or CTS.

The ECI (CIS)-to-WGS 84 ECEF transformation makes use of the new theories of precession [A.1] [A.2] and astronomic nutation [A.3] [A.4] [A.5], the change to a new Standard Epoch (J2000.0) [A.2], and the new definition of universal time [A.6] adopted by the International Astronomical Union (IAU) [A.7].

In the discussion that follows, all coordinate systems are right-handed and orthogonal, and positive rotations are counterclockwise. Such a rotation, about the Z-axis, for example, through an angle α, is designated $R_Z(\alpha)$.

2. Time and Epochs

The two time systems of interest here, sidereal time and universal or solar time, are both based on the diurnal rotation of the earth [A.8]. Sidereal time is usually determined by observing the transits of stars across the observer's meridian. However, since the meridian is involved, consideration must be given to the effect of polar motion on the meridian's position. Mean solar time is associated with a mean or fictitious sun that moves along the celestial equator with a uniform sidereal motion approximately commensurate with the mean rate of the annual motion of the true sun along the ecliptic. If the hour angle of the mean sun is referred to the BIH-defined Zero Meridian, the resulting time is called universal time (UT). Universal and sidereal time are both affected by irregularities in the rotation of the earth. These irregularities take the form of polar motion (variations in the position of the earth's rotation axis with respect to the earth's crust), and variations in the earth's angular rotation rate.

Since 1 January 1972, the time scale distributed by most broadcast time services is coordinated universal time (UTC). It is based on the redefined coordinated universal time [A.9], which differs from another time scale, International Atomic Time (TAI), by an integral number of seconds. The latter is the most precisely determined time scale available, and results from analyses by the BIH of data from the atomic time standards of many countries.

Universal time, determined from observations of the diurnal motions of the stars, and containing non-uniformities due to variations in the rotation of the earth, is called UT0. To obtain UT1, a time scale independent of the observer's position, the effect of the variation of the observer's meridian due to the observed motion of the pole is removed from UT0. The introduction of one second time steps (leap seconds) when necessary, normally at the end of June or December, maintains UTC within 0.90 second of UT1.

Universal time and sidereal time are equivalent time systems since UT1 is formally defined by an equation which relates it to mean sidereal time (MST). The latter is directly obtained from the apparent right ascensions of transiting stars. Therefore, UT1, which forms the basis for the worldwide system of civil time, is derived indirectly from the transit time of stars.

On any given date, a star's computed apparent place, upon which its contribution to the UT1 determination depends, is a function not only of its' catalog position and proper motion, but also of the adopted constants of precession, astronomic nutation, aberration, etc. [A.7]. Therefore, a change from the CIS-FK4 System to the CIS-FK5 System [A.10], and the change in astronomical constants [A.7], have complex and subtle effects on values of UT1. For the new FK5 System, UT1 is defined so as to maintain continuity in value and rate at the epoch of change (1 January 1984) and to be consistent with the origin of the FK5 System. The result is the equation for Greenwich Mean Sidereal Time (GMST) of 0^h UT1 given under the sidereal time transformation (Section 3.4 and Figure A.11, this Appendix). This equation implies that the ratio of solar to sidereal time is 0.997269566329084 at Epoch J2000.0, the inverse of which is 1.002737909350795 [A.7]. The expression also implies a specific angular rotation velocity for the earth, which is discussed later (Section 3.4 and Figure A.11, this Appendix).

The unit of time, T, in the formulas for precession and astronomic nutation is the Julian Century of 36525 days. Since the Julian Day begins at noon, the time interval in the CIS-to-WGS 84 transformation to be discussed later is Julian Centuries measured from 2000 January 1.5.

The conventional relationship between Julian Epochs is [A.2]:

$$JE = 2000.0 + (JED - 2451545.0)/365.25 \qquad \text{(A-1)}$$

where

JE = Julian Epoch

JED = Julian Ephemeris Date.

The relationship between the discontinued Standard Besselian Epoch (BE), B1950.0, and the new Standard Julian Epoch is expressed as [A.2]:

$$BE = 1900.0 + (JED - 2415020.31352)/365.242198781 \qquad \text{(A-2)}$$

The correspondence between six different Julian and Besselian Epochs of interest were calculated [A.2] using Equation (A-2). The numerical results are listed in Table A.1 along with the matrix, M (B1950.0, J2000.0), for transforming (precessing) position and velocity components from the discontinued Standard Besselian Epoch (B1950.0) to the new Standard Julian Epoch (J2000.0) [A.9]. The relevant equation in vector form is [A.9]:

$$\begin{bmatrix} \bar{R} \\ \dot{\bar{R}} \end{bmatrix}_{J2000.0} = M\ (B1950.0,\ J2000.0) \begin{bmatrix} \bar{R} \\ \dot{\bar{R}} \end{bmatrix}_{B1950.0} \qquad \text{(A-3)}$$

where $\bar{R}_{J2000.0}$, $\dot{\bar{R}}_{J2000.0}$ and $\bar{R}_{B1950.0}$, $\dot{\bar{R}}_{B1950.0}$ represent position and velocity components $(X, Y, Z; \dot{X}, \dot{Y}, \dot{Z})$ with respect to Epochs J2000.0 and B1950.0, respectively.

3. Description of CIS (ECI)-to-WGS 84 (ECEF) Transformation Matrices and Equations

3.1 General

In order to accomplish the transformation of position components from CIS (ECI)-to-WGS 84 (ECEF), it is necessary to form the four matrices given in Figure A.1. These are the precession matrix (D), the astronomic nutation matrix (C), the sidereal time matrix (B), and the polar motion matrix (A). The product (ABCD) transforms ECI position components to WGS 84 ECEF position components.

For the transformation of velocity components from CIS (ECI)-to-WGS 84 (ECEF), the rate of change of each of the four matrices must be considered. It is assumed here that the rate of change of the B matrix is the only one that is significant. Therefore, the $\dot{B}$ matrix must be formed and used as shown in Figure A.1 when transforming velocity components from ECI to WGS 84 ECEF.

3.2 Precession (D)

The complex motion of general precession can be specified by three angles ζ, z, and θ. The precession matrix (D) transforms coordinates from the CIS (mean inertial system of epoch) or ECI System to the mean inertial system of date. The coordinate axes for the CIS (mean inertial system of epoch) are designated in Figure A.2 as X_1, Y_1, and Z_1. The X_1-axis is a vector in the plane of the mean celestial equator of epoch pointing toward the mean vernal equinox of epoch ($\bar{\gamma}_0$). The positive end of the X_1-axis is upwards, perpendicular to the page in Figure A.2. The Z_1-axis is a vector perpendicular to the plane of the mean celestial equator of epoch, positive toward the north celestial pole ($\bar{P}_0$). The Y_1-axis is in the plane of the mean celestial equator of epoch, completing a right-handed orthogonal coordinate system (90 degrees east of X_1). The epoch is J2000.0, which is at noon on 1 January in the year 2000.

Precession actually consists of three rotations through three angles:

- A positive rotation about the Z_1-axis through the angle $(90-\zeta)$
- A positive rotation about the X_1'-axis through the angle (θ)
- A negative rotation about the Z_2-axis through the angle $(90+z)$.

The D matrix, as shown in Figure A.3, is the product of these three rotation matrices in the order given.

Equations to compute the three precession angles of ζ, z, and θ are shown in Figure A.4. As given, the units of the computed angles are arc seconds. The time argument, T, is Julian Centuries from J2000.0. The number JED must contain the fractional part of the day required to reach the time of interest within the day. Remember that the Julian Date (JD) begins at noon, so:

$$JED = JD - 0.5 + \text{fractional part of the day.} \tag{A-4}$$

After the precession matrix has been applied, the axes are at the position of the X_2, Y_2, Z_2 axes (mean inertial system of date) as shown in Figure A.2. Here, the X_2-axis is a vector in the plane of the mean equator of date, at the intersection of the mean equator of date and the ecliptic of date pointing toward the mean vernal equinox of date ($\overline{\Upsilon}$). In Figure A.2, the Z_2-axis is a vector perpendicular to the plane of the mean equator of date, positive toward north (or $\overline{P}$). The Y_2-axis is in the plane of the mean equator of date, completing a right-handed orthogonal coordinate system (90 degrees east of X_2).

In vector form:

$$\overline{R}_2 = D\overline{R}_1. \tag{A-5}$$

And, since the D matrix is orthogonal:

$$\overline{R}_1 = D^T\overline{R}_2. \tag{A-6}$$

3.3 Astronomic Nutation (C)

The astronomic nutation matrix (C) transforms coordinates from the mean inertial system of date to the true inertial system of date. This is resolved into nutation-induced corrections known as nutation in ecliptic longitude ($\Delta\psi$) and nutation in obliquity ($\Delta\varepsilon$). The X_2, Y_2, Z_2 axes are rotated to the X_3, Y_3, Z_3 axes.

Nutation consists of three rotations through three angles as described below:

- A positive rotation about the X_2-axis through the angle $(\bar{\varepsilon})$
- A negative rotation about the Z_2'-axis through the angle $(\Delta\psi)$
- A negative rotation about the X_3-axis through the angle (ε) .

These three angles are shown in Figure A.5. The C matrix, as shown in Figure A.6, is the product of these three rotation matrices in the order given.

Equations to compute the three nutation angles of $\bar{\varepsilon}$, $\Delta\psi$, and ε are shown in Figure A.7. The time argument, T, is Julian Centuries from J2000.0. The computation of nutation in longitude $(\Delta\psi)$ and nutation in obliquity $(\Delta\varepsilon)$ both involve a summation over a series of 106 terms. The angles ℓ, ℓ', F, D, and Ω are functions of the position of the sun and moon. The equations to compute these five angles are given in Figure A.8. The units of these angles, as given, are arc seconds. After these angles are changed to units of radians, they are multiplied by the coefficients a_{1i}, a_{2i}, a_{3i}, a_{4i}, and a_{5i} given in Table A.2. The a's are given for each i; the table contains i for 1 through 106. This table also gives the A, B, C, and D terms for each i. The units of A and C are 0.0001 arc second; B and D have units of 0.0001 arc second per Julian Century. Therefore, the units of $\Delta\psi$ and $\Delta\varepsilon$, as given, are 0.0001 arc second. Then, the true obliquity of the ecliptic is the sum of the mean obliquity and the nutation in obliquity.

After the nutation matrix has been applied, the axes are at the position of the X_3, Y_3, Z_3 axes (true inertial system of date). The X_3-axis is in the plane of the CTS Equator (sometimes called the true equator of date), pointing toward the true vernal equinox of date (γ). The positive end of the X_3-axis is upward, perpendicular to the page in Figure A.5. The Z_3-axis is a vector, perpendicular to the plane of the CTS Equator, pointing north toward the Celestial Ephemeris Pole (CEP). The Y_3-axis completes a right-handed orthogonal coordinate system (90 degrees east of X_3).

In vector form:

$$\overline{R}_3 = C\overline{R}_2. \tag{A-7}$$

The C matrix is orthogonal, so:

$$\overline{R}_2 = C^T\overline{R}_3. \tag{A-8}$$

3.4 <u>Sidereal Time (B)</u>

The sidereal time matrix (B) transforms coordinates from the true inertial system of date to true earth-centered, earth-fixed. The X_3, Y_3, Z_3 axes are rotated to the X_4, Y_4, Z_4 axes.

Sidereal time consists of a positive rotation about the Z_3-axis through the angle Λ as shown in Figure A.9. Here, Λ is the longitude of the Zero Meridian from the true vernal equinox of date. In the equation for Λ: H_0 is Greenwich Mean Sidereal Time (GMST) at the beginning of the day; ΔH is apparent minus mean sidereal time or the equation of the equinoxes; ω^* is the rotation rate in a precessing reference frame; t is the UTC time within the day; Δt is UTC minus UT1. The Δt corrects for the irregular rotation of the earth. The quantity ω^* is the sum of the earth's inertial rotation rate and the rate of precession in right ascension of the mean equinox. As the earth rotates to the east, the mean equinox precesses to the west, so the rotation rate used in this transformation must account for both movements.

The B and $\dot{B}$ matrices are shown in Figure A.10.

The equations to compute H_0 and ΔH are given in Figure A.11. Definitions and equations are also given for all other quantities in Λ. Since H_0 is the Greenwich Mean Sidereal Time at 0^h UT1 of JED, the unit of time (T_u) must be designated so as to go to the beginning of the day of interest rather than to noon. In other words, d_u must end in an 0.5 because the Julian day begins at noon.

The value shown for ω^* is consistent with the new definition of universal time and the new theory of precession. In fact, the values for ω^* and ω' are derived from the equation for H_0. To derive the value for ω^*, the rate of precession must be used. The rate of precession, m, is from the new theory of precession. It should be noted that the value given for ω' of $7.2921151467 \times 10^{-5}$ radians per second is consistent with the value given earlier for the earth's inertial rotation rate (Chapter 3, Section 3.3.4).

After the sidereal time matrix has been applied, the axes are at the position of the X_4, Y_4, Z_4 axes. The X_4-axis is in the plane of the CTS Equator, positive toward the Zero Meridian. The Z_4-axis is a vector perpendicular to the plane of the CTS Equator, positive north toward the CEP. The Y_4-axis is in the plane of the CTS Equator, completing a right-handed orthogonal coordinate system (90 degrees east of X_4).

In vector form:

$$\bar{r}_4 = B\bar{r}_3 \qquad \text{(A-9)}$$

$$\bar{r}_3 = B^T\bar{r}_4. \qquad \text{(A-10)}$$

The $\bar{r}_4$ coordinates are affixed to the earth and rotate with it.

3.5 Polar Motion (A)

The polar motion matrix (A) transforms from the true earth-centered, earth-fixed system to a mean earth-centered, earth-fixed system (WGS 84). This motion accounts for the movement of the rotation axis (CEP) with respect to the earth's crust. The X_4, Y_4, Z_4 axes are rotated to the X_5, Y_5, Z_5 axes as shown in Figure A.12.

Polar motion consists of rotations through two angles:

- A negative rotation about the X_4-axis through the angle y_p.

- A negative rotation about the Y_4-axis through the angle x_p.

The quantities x_p and y_p are small angles. Therefore, approximations can be used for the sine and cosine of the angles. The result of making such approximations is shown in Figure A.13 as the A matrix.

After the polar motion matrix has been applied, the axes are at the position of the X_5, Y_5, Z_5 axes (mean earth-centered, earth-fixed coordinate system) as shown in Figure A.12. These axes form the WGS 84 Coordinate System (∃BIH-defined CTS, 1984.0). The X_5-axis is in the plane of the CTS (mean astronomic) Equator, positive toward the Zero Meridian. The Z_5-axis is perpendicular to the CTS Equator, positive north toward the Conventional Terrestrial Pole (CTP) as defined by the BIH on the basis of the coordinates adopted for the BIH stations. The Y_5-axis is in the plane of the CTS Equator, completing a right-handed orthogonal coordinate system (90 degrees east of X_5). (Also, see Chapter 2.)

In vector form:

$$\overline{R}_5 = A\overline{R}_4 \qquad \text{(A-11)}$$

and

$$\overline{R}_4 = A^T \overline{R}_5. \tag{A-12}$$

Both $\overline{R}_4$ and $\overline{R}_5$ coordinates are earth-centered, earth-fixed and rotate with the earth.

4. Summary of Matrices and Equations

All of the matrices previously described are summarized in Figure A.14. The rotations involved in each matrix are also shown. Equations to compute the angles and other quantities required in these matrices are summarized in Figure A.15. The quantities that must be provided are Δt, x_p, and y_p. The only other information needed is tabulated in Table A.2. The steps required to accomplish the ECI (CIS, Epoch J2000.0)-to-WGS 84 (CTS, ECEF) Transformation are summarized in Figure A.16.

The reverse position and velocity transformations, from WGS 84 (CTS, ECEF) to ECI (CIS, Epoch J2000.0), are, respectively:

$$\overline{ECI} = [ABCD]^T \, \overline{ECEF}. \tag{A-13}$$

$$\dot{\overline{ECI}} = [A\dot{B}CD]^T \, \overline{ECEF} + [ABCD]^T \, \dot{\overline{ECEF}}. \tag{A-14}$$

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Table A.1

Correspondence Between Besselian and Julian Epochs
Plus
Matrix for Precessing from B1950.0 to J2000.0

1. Correspondence Between Julian and Besselian Epochs [A.2]

Besselian Epoch	Julian Epoch	JED
B1899.999142	J1900.0	2415020.0
B1900.0	J1900.000858	2415020.31352
B1950.0	J1949.999790	2433282.42345905
B1950.000210	J1950.0	2433282.5
B2000.0	J1999.998723*	2451544.5333981
B2000.001278	J2000.0	2451545.0

2. Matrix For Precessing From B1950.0 to J2000.0 [A.9]

$$M\ (B1950.0,\ J2000.0) =$$

$$\begin{bmatrix} 0.9999256782 & -0.0111820611 & -0.0048579477 & 0.00000242395018 & -0.00000002710663 & -0.00000001177656 \\ 0.0111820610 & 0.9999374784 & -0.0000271765 & 0.00000002710663 & 0.00000242397878 & -0.00000000006587 \\ 0.0048579479 & -0.0000271474 & 0.9999881997 & 0.00000001177656 & -0.00000000006582 & 0.00000242410173 \\ -0.000551 & -0.238565 & 0.435739 & 0.99994704 & -0.01118251 & -0.00485767 \\ 0.238514 & -0.002667 & -0.008541 & 0.01118251 & 0.99995883 & -0.00002718 \\ -0.435623 & 0.012254 & 0.002117 & 0.00485767 & -0.00002714 & 1.00000956 \end{bmatrix}$$

* The last digit given in [A.2] for this value is a misprint.

Table A.2
1980 IAU Theory of Nutation
-Series for Nutation in Longitude ($\Delta\psi$) and Obliquity ($\Delta\varepsilon$) (Mean Equator and Equinox of Date) -

i	a_1	a_2	a_3	a_4	a_5	A	B	C	D
1	0	0	0	0	1	-171996	-174.2	92025	8.9
2	0	0	0	0	2	2062	0.2	-895	0.5
3	-2	0	2	0	1	46	0.0	-24	0.0
4	2	0	-2	0	0	11	0.0	0	0.0
5	-2	0	2	0	2	-3	0.0	1	0.0
6	1	-1	0	-1	0	-3	0.0	0	0.0
7	0	-2	2	-2	1	-2	0.0	1	0.0
8	2	0	-2	0	1	1	0.0	0	0.0
9	0	0	2	-2	2	-13187	-1.6	5736	-3.1
10	0	1	0	0	0	1426	-3.4	54	-0.1
11	0	1	2	-2	2	-517	1.2	224	-0.6
12	0	-1	2	-2	2	217	-0.5	-95	0.3
13	0	0	2	-2	1	129	0.1	-70	0.0
14	2	0	0	-2	0	48	0.0	1	0.0
15	0	0	2	-2	0	-22	0.0	0	0.0
16	0	2	0	0	0	17	-0.1	0	0.0
17	0	1	0	0	1	-15	0.0	9	0.0
18	0	2	2	-2	2	-16	0.1	7	0.0
19	0	-1	0	0	1	-12	0.0	6	0.0
20	-2	0	0	2	1	-6	0.0	3	0.0
21	0	-1	2	-2	1	-5	0.0	3	0.0
22	2	0	0	-2	1	4	0.0	-2	0.0
23	0	1	2	-2	1	4	0.0	-2	0.0
24	1	0	0	-1	0	-4	0.0	0	0.0
25	2	1	0	-2	0	1	0.0	0	0.0
26	0	0	-2	2	1	1	0.0	0	0.0
27	0	1	-2	2	0	-1	0.0	0	0.0
28	0	1	0	0	2	1	0.0	0	0.0
29	-1	0	0	1	1	1	0.0	0	0.0
30	0	1	2	-2	0	-1	0.0	0	0.0
31	0	0	2	0	2	-2274	-0.2	977	-0.5
32	1	0	0	0	0	712	0.1	-7	0.0
33	0	0	2	0	1	-386	-0.4	200	0.0
34	1	0	2	0	2	-301	0.0	129	-0.1
35	1	0	0	-2	0	-158	0.0	-1	0.0
36	-1	0	2	0	2	123	0.0	-53	0.0
37	0	0	0	2	0	63	0.0	-2	0.0
38	1	0	0	0	1	63	0.1	-33	0.0
39	-1	0	0	0	1	-58	-0.1	32	0.0
40	-1	0	2	2	2	-59	0.0	26	0.0
41	1	0	2	0	1	-51	0.0	27	0.0
42	0	0	2	2	2	-38	0.0	16	0.0
43	2	0	0	0	0	29	0.0	-1	0.0
44	1	0	2	-2	2	29	0.0	-12	0.0
45	2	0	2	0	2	-31	0.0	13	0.0
46	0	0	2	0	0	26	0.0	-1	0.0
47	-1	0	2	0	1	21	0.0	-10	0.0
48	-1	0	0	2	1	16	0.0	-8	0.0
49	1	0	0	-2	1	-13	0.0	7	0.0
50	-1	0	2	2	1	-10	0.0	5	0.0
51	1	1	0	-2	0	-7	0.0	0	0.0
52	0	1	2	0	2	7	0.0	-3	0.0
53	0	-1	2	0	2	-7	0.0	3	0.0
54	1	0	2	2	2	-8	0.0	3	0.0
55	1	0	0	2	0	6	0.0	0	0.0
56	2	0	2	-2	2	6	0.0	-3	0.0
57	0	0	0	2	1	-6	0.0	3	0.0
58	0	0	2	2	1	-7	0.0	3	0.0
59	1	0	2	-2	1	6	0.0	-3	0.0
60	0	0	0	-2	1	-5	0.0	3	0.0
61	1	-1	0	0	0	5	0.0	0	0.0
62	2	0	2	0	1	-5	0.0	3	0.0
63	0	1	0	-2	0	-4	0.0	0	0.0
64	1	0	-2	0	0	4	0.0	0	0.0
65	0	0	0	1	0	-4	0.0	0	0.0
66	1	1	0	0	0	-3	0.0	0	0.0
67	1	0	2	0	0	3	0.0	0	0.0
68	1	-1	2	0	2	-3	0.0	1	0.0
69	-1	-1	2	2	2	-3	0.0	1	0.0
70	-2	0	0	0	1	-2	0.0	1	0.0
71	3	0	2	0	2	-3	0.0	1	0.0
72	0	-1	2	2	2	-3	0.0	1	0.0
73	1	1	2	0	2	2	0.0	-1	0.0
74	-1	0	2	-2	1	-2	0.0	1	0.0
75	2	0	0	0	1	2	0.0	-1	0.0
76	1	0	0	0	2	-2	0.0	1	0.0
77	3	0	0	0	0	2	0.0	0	0.0
78	0	0	2	1	2	2	0.0	-1	0.0
79	-1	0	0	0	2	1	0.0	-1	0.0
80	1	0	0	-4	0	-1	0.0	0	0.0
81	-2	0	2	2	2	1	0.0	-1	0.0
82	-1	0	2	4	2	-2	0.0	1	0.0
83	2	0	0	-4	0	-1	0.0	0	0.0
84	1	1	2	-2	2	1	0.0	-1	0.0
85	1	0	2	2	1	-1	0.0	1	0.0
86	-2	0	2	4	2	-1	0.0	1	0.0
87	-1	0	4	0	2	1	0.0	0	0.0
88	1	-1	0	-2	0	1	0.0	0	0.0
89	2	0	2	-2	1	1	0.0	-1	0.0
90	2	0	2	2	2	-1	0.0	0	0.0
91	1	0	0	2	1	-1	0.0	0	0.0
92	0	0	4	-2	2	1	0.0	0	0.0
93	3	0	2	-2	2	1	0.0	0	0.0
94	1	0	2	-2	0	-1	0.0	0	0.0
95	0	1	2	0	1	1	0.0	0	0.0
96	-1	-1	0	2	1	1	0.0	0	0.0
97	0	0	-2	0	1	-1	0.0	0	0.0
98	0	0	2	-1	2	-1	0.0	0	0.0
99	0	1	0	2	0	-1	0.0	0	0.0
100	1	0	-2	-2	0	-1	0.0	0	0.0
101	0	-1	2	0	1	-1	0.0	0	0.0
102	1	1	0	-2	1	-1	0.0	0	0.0
103	1	0	-2	2	0	-1	0.0	0	0.0
104	2	0	0	2	0	1	0.0	0	0.0
105	0	0	2	4	2	-1	0.0	0	0.0
106	0	1	0	1	0	1	0.0	0	0.0

Units: A = C = 0.0001"; B = D = 0.0001" Per Julian Century (T from Epoch J2000.0)

Position

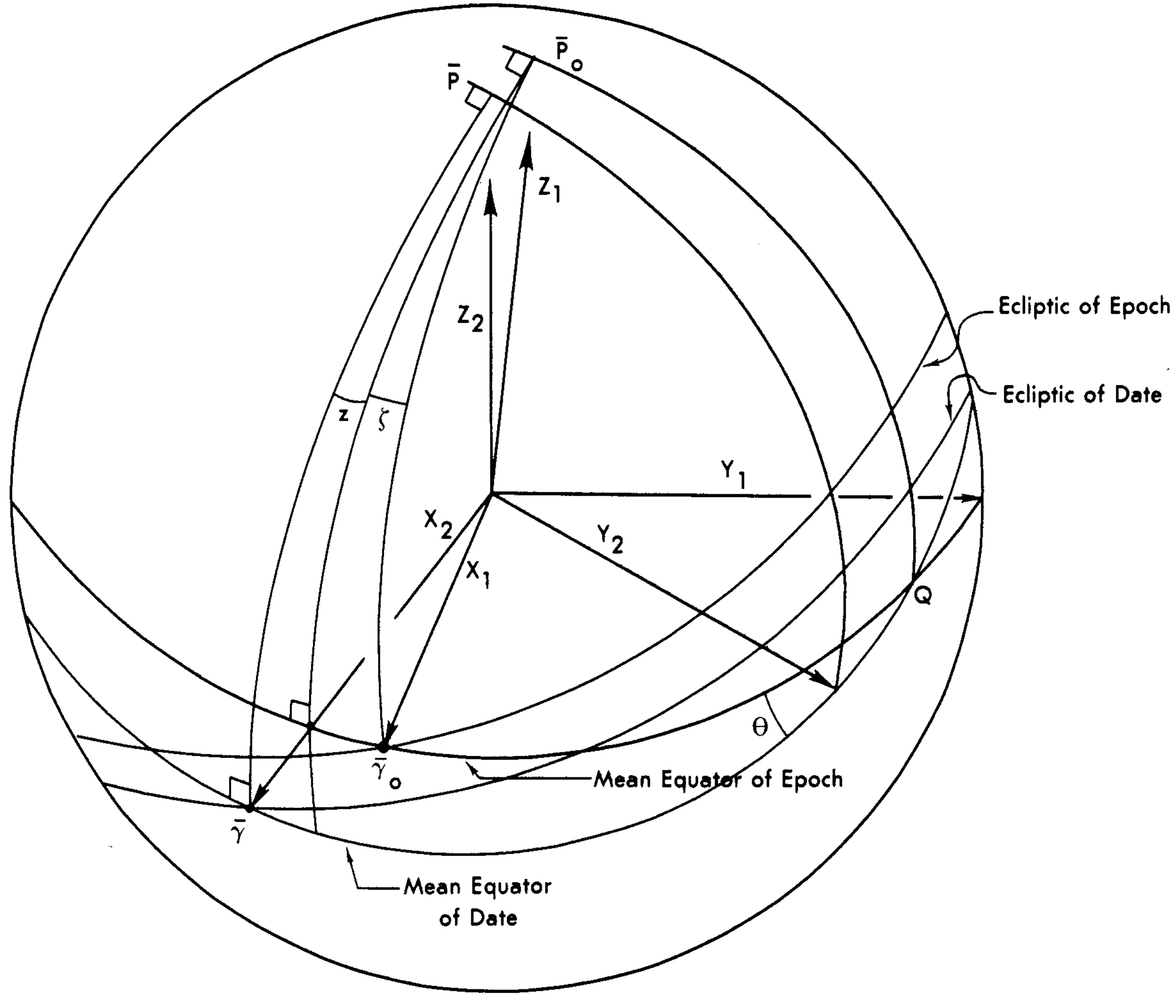
$$\overline{ECEF} = [ABCD]\,\overline{ECI}$$

Velocity

$$\dot{\overline{ECEF}} = [A\dot{B}CD]\,\overline{ECI} + [ABCD]\,\dot{\overline{ECI}}$$

ECI Epoch = J2000.0
J = Julian
ECEF = Earth-Centered, Earth-Fixed
ECI = Earth-Centered, Inertial
D = Precession
C = Astronomic Nutation
B = Sidereal Time
A = Polar Motion
$\dot{B}$ = Rate of Change of the B Matrix (With Time)

Figure A.1. ECI (CIS, Epoch J2000.0)-to-WGS 84 (CTS, ECEF) Transformation



ζ, θ, z = Precession Parameters

$\bar{\gamma}_o$ to Q = $90° - \zeta$

θ = Angle Between Equators

Q to $\bar{\gamma}$ = $-(90° + z)$

Figure A.2. Celestial Sphere Depicting Precession of the Equinox

$$D = R_z[-(90+z)]\ R_x[\theta]\ R_z[90-\zeta]$$

$$D = \begin{bmatrix} \cos z \cos\theta \cos\zeta - \sin z \sin\zeta & -\cos z \cos\theta \sin\zeta - \sin z \cos\zeta & -\cos z \sin\theta \\ \sin z \cos\theta \cos\zeta + \cos z \sin\zeta & -\sin z \cos\theta \sin\zeta + \cos z \cos\zeta & -\sin z \sin\theta \\ \sin\theta \cos\zeta & -\sin\theta \sin\zeta & \cos\theta \end{bmatrix}$$

Figure A.3. Precession Transformation Matrix

$$\zeta = 2306.2181\ T + 0.30188\ T^2 + 0.017998\ T^3$$

$$z = 2306.2181\ T + 1.09468\ T^2 + 0.018203\ T^3$$

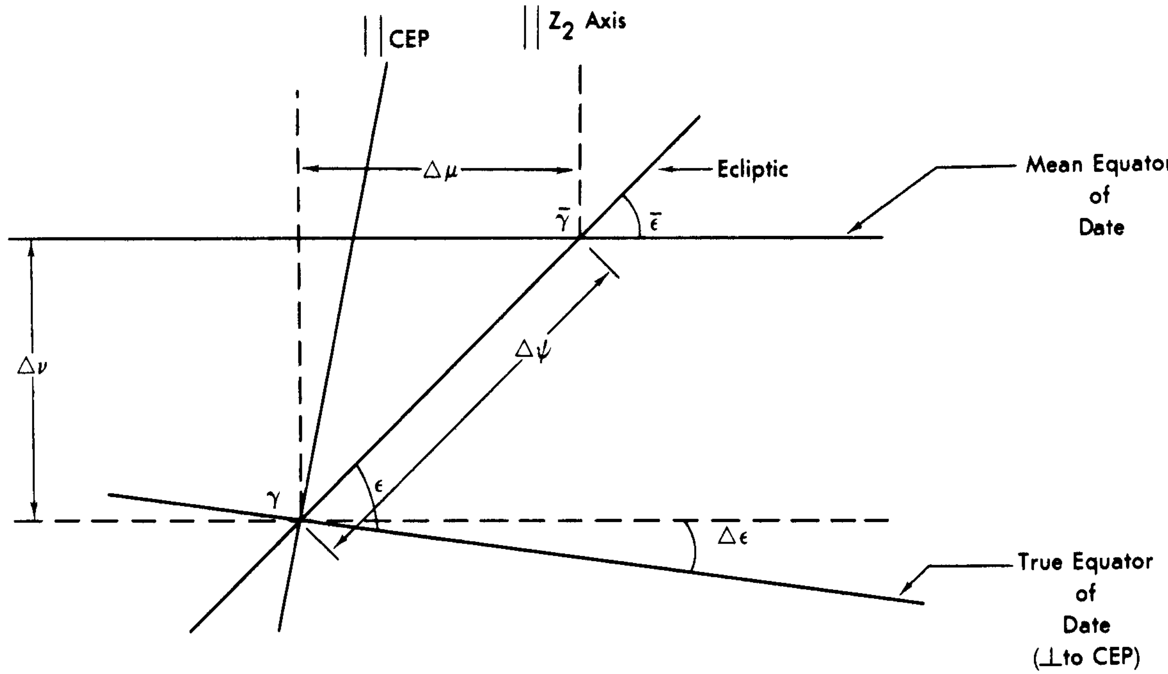
$$\theta = 2004.3109\ T - 0.42665\ T^2 - 0.041833\ T^3$$

Arc Seconds [A.2] [A.7] [A.9]
[These Three Equations Obtained by Setting T=0 in Equation (7) of Reference A.2, Where T has a Different Meaning Than Here.]

$T = [JED - 2451545]/36525$
= Julian Centuries from Epoch J2000.0 (2000 Jan 1.5)

JED = Julian Ephemeris Date

Figure A.4. Precession Equations



$\Delta\mu$ = Nutation in Right Ascension

$\Delta\nu$ = Nutation in Declination

$\tan\ \Delta\mu = \tan \Delta\psi\ \cos \bar{\epsilon}$

$\sin\ \Delta\nu = \sin \Delta\psi\ \sin \bar{\epsilon}$

Nutation Parameters $\bar{\epsilon}$, $\Delta\psi$, ϵ :

$\bar{\epsilon}$ = Mean Obliquity of Ecliptic

$\Delta\psi$ = Nutation in Longitude

$\epsilon = \bar{\epsilon} + \Delta\epsilon$ = True Obliquity of Ecliptic

$\Delta\epsilon$ = Nutation in Obliquity

Figure A.5. Astronomic Nutation Diagram

$$C = R_x[-\varepsilon]\, R_z[-\Delta\psi]\, R_x[\,\bar{\varepsilon}\,]$$

$$C = \begin{bmatrix} \cos\Delta\psi & -\sin\Delta\psi \cos\bar{\varepsilon} & -\sin\Delta\psi \sin\bar{\varepsilon} \\ \cos\varepsilon \sin\Delta\psi & \cos\varepsilon \cos\Delta\psi \cos\bar{\varepsilon} + \sin\varepsilon \sin\bar{\varepsilon} & \cos\varepsilon \cos\Delta\psi \sin\bar{\varepsilon} - \sin\varepsilon \cos\bar{\varepsilon} \\ \sin\varepsilon \sin\Delta\psi & \sin\varepsilon \cos\Delta\psi \cos\bar{\varepsilon} - \cos\varepsilon \sin\bar{\varepsilon} & \sin\varepsilon \cos\Delta\psi \sin\bar{\varepsilon} + \cos\varepsilon \cos\bar{\varepsilon} \end{bmatrix}$$

Figure A.6. Astronomic Nutation Transformation Matrix

$$\bar{\varepsilon} = \varepsilon_0 - 46.8150\ T - 0.00059\ T^2 + 0.001813\ T^3 \quad \text{Arc Seconds}$$

$$= \text{Mean Obliquity of Ecliptic}$$

$$\varepsilon_0 = 23^0\ 26'\ 21.448'' = 84381.448''$$

$$\Delta\psi = \sum_{i=1}^{106} \Delta\psi_i = \text{Nutation in Longitude}$$

$$\Delta\psi_i = (A_i + B_i T) \sin(a_{1i}\ell + a_{2i}\ell' + a_{3i}F + a_{4i}D + a_{5i}\Omega)$$

$$\Delta\varepsilon = \sum_{i=1}^{106} \Delta\varepsilon_i = \text{Nutation in Obliquity}$$

$$\Delta\varepsilon_i = (C_i + D_i T) \cos(a_{1i}\ell + a_{2i}\ell' + a_{3i}F + a_{4i}D + a_{5i}\Omega)$$

$$\varepsilon = \bar{\varepsilon} + \Delta\varepsilon$$

$$= \text{True Obliquity of Ecliptic}$$

$$T = [\text{JED} - 2451545]/36525 = d/36525$$
$$= \text{Julian Centuries from Epoch J2000.0}$$

Figure A.7. Astronomic Nutation Equations (1980 IAU Theory)

$$\ell = 485866.733 + (1325^r + 715922.633)T + 31.310T^2 + 0.064T^3 \quad \text{Arc Seconds}$$

= Mean Anomaly of Moon

$$\ell' = 1287099.804 + (99^r + 1292581.244)T - 0.577T^2 - 0.012T^3 \quad \text{Arc Seconds}$$

= Mean Anomaly of Sun

$$F = 335778.877 + (1342^r + 295263.137)T - 13.257T^2 + 0.011T^3 \quad \text{Arc Seconds}$$

= (Mean Longitude of Moon) - Ω

$$D = 1072261.307 + (1236^r + 1105601.328)T - 6.891T^2 + 0.019T^3 \quad \text{Arc Seconds}$$

= Mean Elongation of Moon From Sun

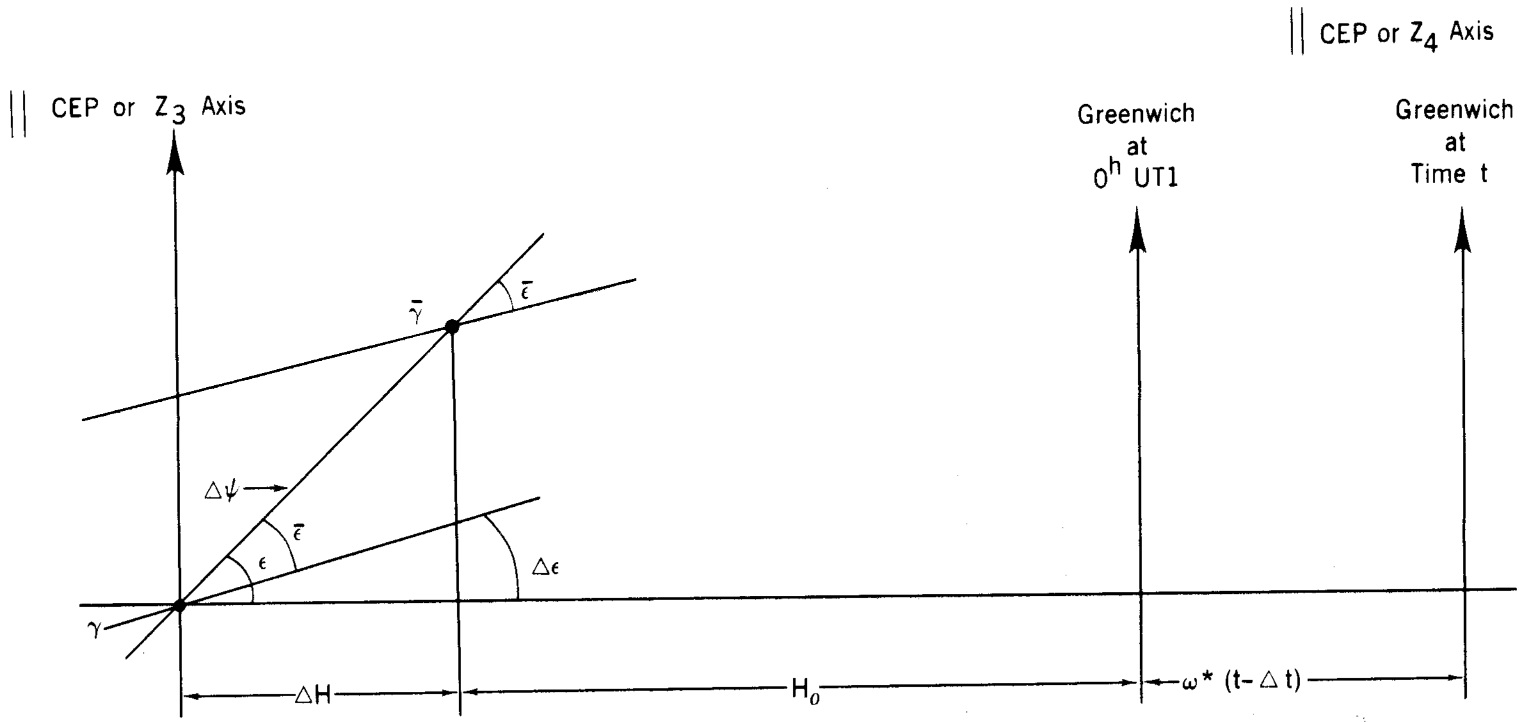
$$\Omega = 450160.280 - (5^r + 482890.539)T + 7.455T^2 + 0.008T^3 \quad \text{Arc Seconds}$$

= Longitude of Ascending Node of Lunar Mean Orbit on Ecliptic Measured From Mean Equinox of Date

$$T = (JED - 2451545)/36525$$

$$1^r = 1296000''$$

Figure A.8. Astronomic Nutation Arguments



$$\Lambda = H_o + \Delta H + \omega^* (t - \Delta t)$$

$$\omega^* = \omega' + m$$

ω^* = Rotation Rate in Precessing Reference Frame
ω' = Earth's Inertial Rotation Rate
m = Rate of Precession in Right Ascension

Figure A.9. Sidereal Time Diagram

$$B = R_z[\Lambda]$$

$$B = \begin{bmatrix} \cos\Lambda & \sin\Lambda & 0 \\ -\sin\Lambda & \cos\Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{B} = \begin{bmatrix} -\omega^*\sin\Lambda & \omega^*\cos\Lambda & 0 \\ -\omega^*\cos\Lambda & -\omega^*\sin\Lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure A.10. Sidereal Time Transformation Matrices

$$\Lambda = H_0 + \Delta H + \omega^*(t - \Delta t)$$

$$H_0 = 24110.54841 + 8640184.812866\ T_u + 0.093104\ T_u^2 - 6.2 \times 10^{-6}\ T_u^3 \text{ Seconds of Time}$$

= Greenwich Mean Sidereal Time at 0^h UT1 of JED

$$T_u = d_u/36525; \quad d_u = \text{JED} - 2451545; \quad d_u \rightarrow \pm 0.5, 1.5, 2.5, \ldots$$

$$\Delta H = \arctan(\cos\varepsilon \tan\Delta\psi)$$

= (Apparent Minus Mean) Sidereal Time

ε = True Obliquity; $\Delta\psi$ = Nutation in Longitude

t = Time Within Day (UTC); Δt = UTC - UT1

ω^* = Rotation Rate in Precessing Reference Frame

$= \omega' + m = 7.2921158553 \times 10^{-5} + 4.3 \times 10^{-15}\ T_u$ (Radians/Second)

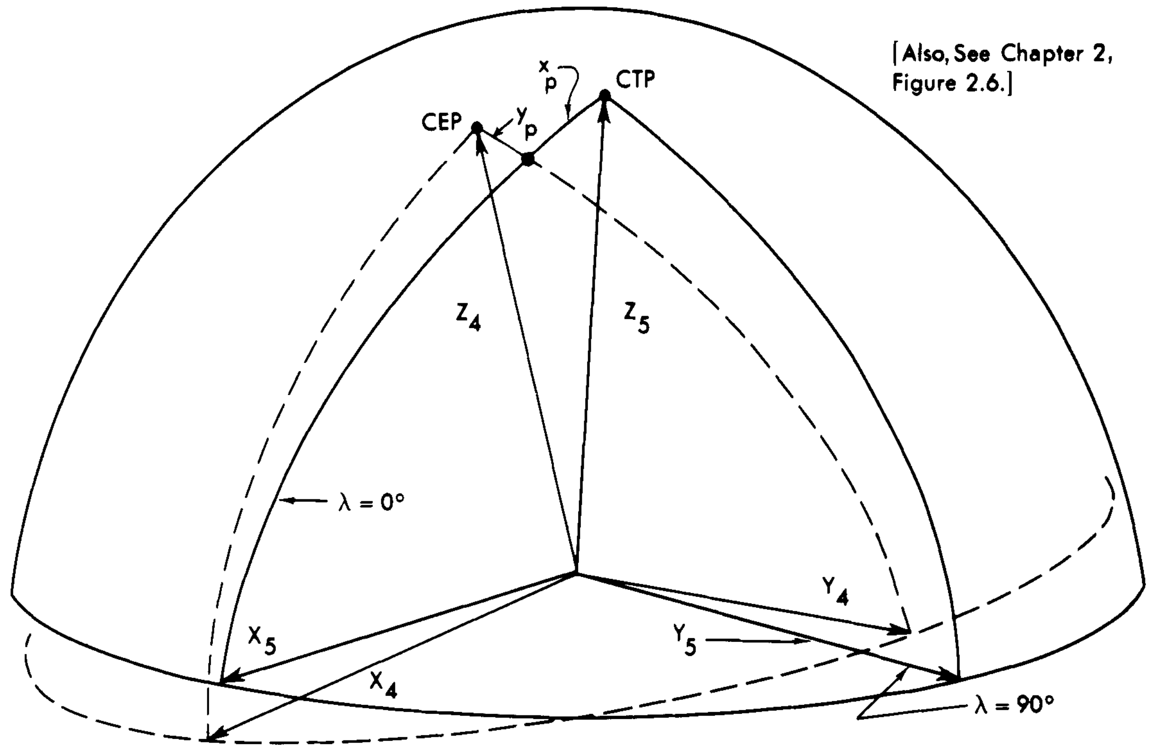
ω' = Earth's Inertial Rotation Rate

$= 7.2921151467 \times 10^{-5}$ (Radians/Second)

m = Rate of Precession in Right Ascension

$= 7.086 \times 10^{-12} + 4.3 \times 10^{-15}\ T_u$ (Radians/Second)

Figure A.11. Sidereal Time Equations



CTP = Conventional Terrestrial Pole
CEP = Celestial Ephemeris Pole

Figure A.12. Polar Motion Diagram

$$A = R_y\,[-x_p]\;R_x\,[-y_p]$$

[Since x_p and y_p are small angles, it is technically permissible to use approximate transformation matrix]

$$A \cong \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{bmatrix}$$

x_p = Angular displacement of CEP from mean terrestrial pole measured along Zero Meridian (positive south)

y_p = Angular displacement of CEP from mean terrestrial pole measured normal to Zero Meridian (positive west)

Figure A.13. Polar Motion Transformation Matrix

$$D = R_z[-(90+z)]\ R_x[\theta]\ R_z[90-\zeta]$$

$$D = \begin{bmatrix} \cos z \cos\theta \cos\zeta - \sin z \sin\zeta & -\cos z \cos\theta \sin\zeta - \sin z \cos\zeta & -\cos z \sin\theta \\ \sin z \cos\theta \cos\zeta + \cos z \sin\zeta & -\sin z \cos\theta \sin\zeta + \cos z \cos\zeta & -\sin z \sin\theta \\ \sin\theta \cos\zeta & -\sin\theta \sin\zeta & \cos\theta \end{bmatrix}$$

$$C = R_x\ [-\varepsilon]\ R_z[-\Delta\psi]\ R_x[\ \bar{\varepsilon}\ ]$$

$$C = \begin{bmatrix} \cos\Delta\psi & -\sin\Delta\psi \cos\bar{\varepsilon} & -\sin\Delta\psi \sin\bar{\varepsilon} \\ \cos\varepsilon \sin\Delta\psi & \cos\varepsilon \cos\Delta\psi \cos\bar{\varepsilon} + \sin\varepsilon \sin\bar{\varepsilon} & \cos\varepsilon \cos\Delta\psi \sin\bar{\varepsilon} - \sin\varepsilon \cos\bar{\varepsilon} \\ \sin\varepsilon \sin\Delta\psi & \sin\varepsilon \cos\Delta\psi \cos\bar{\varepsilon} - \cos\varepsilon \sin\bar{\varepsilon} & \sin\varepsilon \cos\Delta\psi \sin\bar{\varepsilon} + \cos\varepsilon \cos\bar{\varepsilon} \end{bmatrix}$$

$$B = R_z[\Lambda]$$

$$B = \begin{bmatrix} \cos\Lambda & \sin\Lambda & 0 \\ -\sin\Lambda & \cos\Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dot{B} = \begin{bmatrix} -\omega^*\sin\Lambda & \omega^*\cos\Lambda & 0 \\ -\omega^*\cos\Lambda & -\omega^*\sin\Lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = R_y[-x_p]\ R_x[-y_p]$$

$$A \cong \begin{bmatrix} 1 & 0 & x_p \\ 0 & 1 & -y_p \\ -x_p & y_p & 1 \end{bmatrix}$$

$$\overline{ECEF} = [ABCD]\ \overline{ECI}\ ; \qquad \dot{\overline{ECEF}} = [A\dot{B}CD]\ \overline{ECI} + [ABCD]\ \dot{\overline{ECI}}$$

Figure A.14. Summary - ECI (CIS, Epoch J2000.0)-to-WGS 84 (CTS, ECEF) Transformation Matrices

Time Interval From J2000.0

$$d = JED - 2451545.0; \; T = d/36525; \; JED = \text{Julian Ephemeris Date}$$

[d = Julian Ephemeris Days;
T = Julian Centuries;
Both From J2000.0]

Precession

$$\zeta = 2306.2181T + 0.30188T^2 + 0.017998T^3$$
$$z = 2306.2181T + 1.09468T^2 + 0.018203T^3$$
$$\theta = 2004.3109T - 0.42665T^2 - 0.041833T^3$$

(Seconds of Arc)

Nutation

$$\bar{\varepsilon} = 84381.448 - 46.8150T - 0.00059T^2 + 0.001813T^3 \qquad \text{(Seconds of Arc)}$$

$$\Delta\psi_i = (A_i + B_i T)\sin(a_{1i}\ell + a_{2i}\ell' + a_{3i}F + a_{4i}D + a_{5i}\Omega)$$
$$\Delta\varepsilon_i = (C_i + D_i T)\cos(a_{1i}\ell + a_{2i}\ell' + a_{3i}F + a_{4i}D + a_{5i}\Omega)$$

$$\Delta\psi = \sum_{i=1}^{106} \Delta\psi_i; \quad \Delta\varepsilon = \sum_{i=1}^{106} \Delta\varepsilon_i; \; \text{(Units = 0.0001 Arc Second)}$$

$$\varepsilon = \bar{\varepsilon} + \Delta\varepsilon; \; \text{(Units = Seconds of Arc)}$$

$$\ell = 485866.733 + (1325^r + 715922.633)T + 31.310T^2 + 0.064T^3$$
$$\ell' = 1287099.804 + (99^r + 1292581.244)T - 0.577T^2 - 0.012T^3$$
$$F = 335778.877 + (1342^r + 295263.137)T - 13.257T^2 + 0.011T^3$$
$$D = 1072261.307 + (1236^r + 1105601.328)T - 6.891T^2 + 0.019T^3$$
$$\Omega = 450160.280 - (5^r + 482890.539)T + 7.455T^2 + 0.008T^3$$

(Seconds of Arc)

(1^r = 1296000")

Sidereal Time

$$\Lambda = H_0 + \Delta H + \omega^*(t - \Delta t), \; \text{(Units = Seconds)}; \; \Delta H = \arctan(\cos\varepsilon \tan\Delta\psi), \; \text{(Units = Radians; Must be Changed to Seconds)}$$

$$H_0 = 24110.54841 + 8640184.812866T_u + 0.093104T_u^2 - 6.2\times10^{-6}\,T_u^3 \qquad \text{(Seconds)}$$

$$t = \text{time within day (UTC)}; \quad \Delta t = UTC - UT1 \qquad \text{(Seconds)}$$

$$\omega^* = 7.2921158553\times10^{-5} + 4.3\times10^{-15}\,T_u \qquad \text{(Radians/Second)}$$

$$T_u = d_u/36525; \quad d_u = JED - 2451545 \text{ at } 0^h \text{ UT1}; \quad d_u \rightarrow \pm0.5, 1.5, 2.5, \ldots$$

Figure A.15. Summary - ECI (CIS, Epoch J2000.0)-to-WGS 84 (CTS, ECEF) Equations

Coordinate System	Transformation Matrix	Position Vector	Velocity Vector
Mean Inertial of J2000.0 or Conventional Inertial (CIS)		$\overline{R}_1$ or $\overline{ECI}$	$\dot{\overline{R}}_1$ or $\dot{\overline{ECI}}$
	Precession (D)		
Mean Earth-Centered Inertial of Date		$\overline{R}_2 = D\ \overline{R}_1$	$\dot{\overline{R}}_2 = D\ \dot{\overline{R}}_1$
	Astronomic Nutation (C)		
True Earth-Centered Inertial of Date		$\overline{R}_3 = C\ \overline{R}_2$	$\dot{\overline{R}}_3 = C\ \dot{\overline{R}}_2$
	Sidereal Time (B)		
True Earth-Centered, Earth-Fixed		$\overline{R}_4 = B\ \overline{R}_3$	$\dot{\overline{R}}_4 = B\ \dot{\overline{R}}_3 + \dot{B}\ \overline{R}_3$
	Polar Motion (A)		
Mean Earth-Centered, Earth-Fixed or Conventional Terrestrial (CTS)		$\overline{R}_5 = A\ \overline{R}_4 = \overline{ECEF}$	$\dot{\overline{R}}_5 = A\ \dot{\overline{R}}_4 = \dot{\overline{ECEF}}$

Figure A.16. Summary - Steps in ECI (CIS, Epoch J2000.0)-to-WGS 84 (CTS, ECEF) Transformation

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